A bijection which implies Melzer's polynomial identities: the $\chi_{1,1}^{(p,p+1)}$ case

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Abstract

We obtain a bijection between certain lattice paths and partitions. This implies a proof of polynomial identities conjectured by Melzer. In a limit, these identities reduce to Rogers–Ramanujan-type identities for the $\chi_{1,1}^{(p,p+1)}(q)$ Virasoro characters, conjectured by the Stony Brook group.

1   Introduction

In [1], Melzer conjectured polynomial identities that include the following:

$$
\sum_{m_1,\ldots,m_{p-2}=0}^{\infty} q^{\vec{m}^T C_{p-2}\,\vec{m}} \prod_{j=1}^{p-2} \begin{bmatrix} m_{j-1}+m_{j+1} \\ 2m_j \end{bmatrix}_q
$$
$$
= \sum_{j=-\infty}^{\infty} \left\{ q^{p(p+1)j^2+j} \begin{bmatrix} 2m \\ m-(p+1)j \end{bmatrix}_q - q^{p(p+1)j^2+(2p+1)j+1} \begin{bmatrix} 2m \\ m-1-(p+1)j \end{bmatrix}_q \right\}. \quad (1)
$$

Here $m_0 \equiv m$, $m_{p-1} \equiv 0$, $\vec{m}^T = (m_1,\ldots,m_{p-2})$, C_{p-2} is the Cartan matrix of the Lie algebra A_{p-2}:

$$
(C_{p-2})_{j,k} = 2\delta_{j,k} - \delta_{j,k+1} - \delta_{j,k+1}, \qquad j,k = 1,\ldots,p-2, \quad (2)
$$

and $\begin{bmatrix} N \\ m \end{bmatrix}_q$ is the q-binomial coefficient:

$$
\begin{bmatrix} N \\ m \end{bmatrix}_q =
\begin{cases}
\dfrac{(q)_N}{(q)_m (q)_{N-m}} & 0 \le m \le N \\[2ex]
0 & otherwise,
\end{cases}
\quad (3)
$$

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with $(q)_m = \prod_{k=1}^{m}(1 - q^k)$ for $m > 0$ and $(q)_0 = 1$. In the limit $m \to \infty$, the above identities reduce to Rogers–Ramanujan–type identities for the $\chi_{1,1}^{(p,p+1)}(q)$ Virasoro characters, conjectured by the Stony Brook group in [3]. For an introduction to these identities, and further references, we refer the reader to [4].

In [5, 6] proofs of the above identities obtained. In this letter, we present a bijective proof based on the following observation: The LHS of (1) is the generating function of certain *restricted lattice paths* of finite lengths [6]. On the other hand, the RHS is the generating function of partitions with prescribed hook differences [7]. In this work, we establish a bijection between these lattice paths and partitions.

The bijection that we obtain is *so* simple, it almost trivializes the identities (given that we know how to evaluate the generating functions of the combinatoric objects that appear on each side: the lattice paths, and the Ferrers graphs, and this is *not* a trivial task). We hope that this work will stimulate the search for further bijections that equally simply imply the rest of Melzer's and other polynomial identities. Since bijections contain *detailed* information about the objects involved, we hope that this and similar proofs help gain better understanding of the full meaning of character identities.

In §2, we define the lattice paths generated by the RHS of (1), and the partitions generated by the LHS. In §3, we establish a bijection between these lattice paths and partitions, using certain matrices as interpolating structures. §4 contains a short discussion.

2 Paths and partitions

2.1 Admissible lattice paths

We define an *admissible sequence of integers* Σ as an ordered sequence $\{\sigma_0, \sigma_1, \sigma_2, \ldots, \sigma_{2m}\}$, with $|\sigma_j - \sigma_{j+1}| = 1$, for all j, $\sigma_0 = \sigma_{2m} = 0$, and each $\sigma_j \in \{0, 1, \ldots, p-1\}$.

Given Σ, we obtain *an admissible lattice path* $P(\Sigma)$ as follows: plot (j, σ_j), $j = 0, \ldots, 2m$ in R^2, and connect each pair of adjacent vertices (j, σ_j) and $(j+1, \sigma_{j+1})$ by a straight line-segment. An example of P with $p \geq 6$ is shown in Fig. 1. For the rest of this work, the term *path* will always be used to imply an admissible lattice path in the above sense.

2.1.1 Vertices on a path

On a path, we distinguish four different types of vertices:

V1 If $\sigma_{j-1} < \sigma_j < \sigma_{j+1}$, then (j, σ_j) is an up-vertex,

V2 if $\sigma_{j-1} > \sigma_j > \sigma_{j+1}$, then (j, σ_j) is a down-vertex,

V3 if $\sigma_{j-1} < \sigma_j > \sigma_{j+1}$, then (j, σ_j) is a maximum,

V4 if $\sigma_{j-1} > \sigma_j < \sigma_{j+1}$, then (j, σ_j) is a minimum.

By definition, the end-points $(0, 0)$ and $(2m, 0)$ are both minima. Notice that a path consists of a number of sections, each connecting a maximum and an adjacent minimum. Since each path begins and ends on the x-axis, there is always an even number of such sections, with an equal number of *ascending* and *descending* sections, as one scans the paths from one end to the other.

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The total number of these sections is twice the number of maxima. Furthermore, there is always an equal number of up- and down-vertices, and one more minimum than maximum. Example: in Fig. 1, we find 14 sections, 6 up- and down-vertices, 7 maxima and 8 minima adding up to $27 = 2m + 1$.

2.1.2 The weight $W(P)$ of a path

To a path P we assign the *weight $W(P)$* as half the sum of the x-coordinates of its up- and down-vertices. For the example of Fig. 1, we obtain the weight $(1 + 2 + 4 + 6 + 7 + 8 + 10 + 11 + 14 + 15 + 17 + 21)/2 = 58$. The generating function F of our paths is defined as

$$F(q) = \sum_P q^{W(P)}. \tag{4}$$

Theorem [6]: For all $p \in \mathbb{Z}_{\geq 2}$ and $m \in \mathbb{Z}_{\geq 0}$ we have

$$F(q) = \sum_{m_1,\ldots,m_{p-2}=0}^{\infty} q^{\vec{m}^T C_{p-2} \vec{m}} \prod_{j=1}^{p-2} \begin{bmatrix} m_{j-1} + m_{j+1} \\ 2m_j \end{bmatrix}_q, \tag{5}$$

For proof, we refer the reader to [6]. Note that the case $p = 2$ is trivial, as we have only a single path without any up- or down-vertices, and $F = 1$.

2.2 Partitions with prescribed hook differences

For completeness, we list a number of standard definitions from the theory of partitions [2, 7].

A *partition* of $n \in \mathbb{Z}_{>0}$ is a sequence of integers $\{r_1, r_2, \ldots, r_N\}$, with $\sum_j r_j = n$ and $r_1 \geq r_2 \geq \ldots \geq r_N > 0$ [2]. r_1 and N are the largest part and the number of parts of the partition, respectively. Example: a partition of 58 is $11, 9, 9, 8, 8, 7, 4, 1, 1$. Here, the largest part is 11, and the number of parts is 8.

The *Ferrers graph* corresponding to a partition is obtained by drawing N rows of nodes, with the j-th row containing r_j nodes, counting rows from top to bottom. The Ferrers graph of the above partition of 58 is shown in Fig. 2a. The largest square of nodes one can draw in a Ferrers graph is called the *Durfee square*. The dimension D of the side of a Durfee square is the number of $r_j \geq j$.

The *length* of the j-th row/column of a Ferrers graph is the number of nodes of the j-th column/row, counting columns from left to right. The length of the j-th row being r_j, the length of the j-th column will be denoted c_j.

The (j, k)-th node of a Ferrers graph is the node in the j-th row and k-th column. The nodes with coordinates $(k + j, k)$ form the *j-th diagonal*. A quantity that will play an important role is this work is the *hook difference at node (j, k)*, defined as $r_j - c_k$

The Ferrers graph of the above partition of 58 is shown in Fig. 2a. In this same figure, we also show the Durfee square and indicate some of the diagonals.

2.2.1 Admissible partitions

The partitions relevant to (1) satisfy two conditions. In the language of Ferrers graphs, these are:

F1 The 'boundary conditions': the number of columns $\leq m$, and the number of rows is $\leq m - 1$.

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F2 The hook differences: on the zeroth diagonal they satisfy ≥ 1, on the $(p-2)$-th diagonal they satisfy ≤ 0.

For the rest of this work, we use *partition* to mean an admissible partition that satisfies the above conditions. For the partition of Fig. 2a we have computed the hook differences on the $0, 3$ and 4-th diagonal in Fig. 2b. We hence find admissibility for $p \geq 6, m \geq 11$.

The generating function B of our partitions is defined as

$$B(q) = \sum_{n=0}^{\infty} G\left(\Pi(n)\right) q^n, \tag{6}$$

with $G(n)$ the number of partitions of n.

Theorem [7]: For all $p \in \mathbb{Z}_{\geq 2}$ and $m \in \mathbb{Z}_{\geq 0}$ we have

$$B(q) = \sum_{j=-\infty}^{\infty} \left\{ q^{p(p+1)j^2+j} \begin{bmatrix} 2m \\ m - (p+1)j \end{bmatrix}_q - q^{p(p+1)j^2+(2p+1)j+1} \begin{bmatrix} 2m \\ m - 1 - (p+1)j \end{bmatrix}_q \right\}, \tag{7}$$

For proof, we refer the reader to [7].

As for the generating function F of the lattice paths, we note that for $p = 2$ things trivialize. Clearly, from our definition of admissible partitions, for $p = 2$ we obtain $B = 1$, as the restrictions on the 0-th diagonal are exclusive. (The generating function of the partition of 0 is 1, by definition.) To observe that expression (7) indeed yields unity one has to invoke the finite analogue of Euler's identity [2].

3 A bijection between paths and partitions

3.1 Interpolating matrices

Since a path is uniquely defined in terms of the ordered sequence of up- and down-vertices that it contains, we can encode it in terms of a matrix whose elements are precisely such a sequence. Such a matrix will interpolate between a path and the corresponding partition.

3.2 From paths to matrices

A path P can be encoded into a 2–by–$\mathcal{N}$ matrix $M(P) = \begin{pmatrix} u_1 & u_2 & \ldots & u_{\mathcal{N}} \\ d_1 & d_2 & \ldots & d_{\mathcal{N}} \end{pmatrix}$, with $\mathcal{N}$ the number of maxima of P, as follows: u_j is the number of up-vertices between the j-th minimum and the j-th maximum, and d_j is the number of down-vertices between the j-th maximum and the $(j+1)$-th minimum. Here we count minima and maxima from left to right, $(0,0)$ being the first minimum. Example: for the path of Fig. 1, we obtain $M = \begin{pmatrix} 2 & 3 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 & 0 \end{pmatrix}$.

3.2.1 Restrictions on the interpolating matrices

The restrictions on the paths translate into restrictions on the interpolating matrices as follows:

R1 Since a path begins and ends on the real axis, the elements of the corresponding matrix satisfy

$$\sum_{j=1}^{\mathcal{N}} u_j = \sum_{j=1}^{\mathcal{N}} d_j$$

R2 Since the paths have length $2m$, the total number of *bonds* on a path is $2m$. In any section between a maximum and a minimum, the number of bonds exceeds the number of vertices by one. Since the elements of the interpolating matrices are precisely the number of vertices on such sections, they satisfy

$$2\mathcal{N} + \sum_{j=1}^{\mathcal{N}} (u_j + d_j) = 2m$$

R3 The condition that paths are confined to the strip $0 \le y \le p - 1$ gives

$$\begin{aligned}\sum_{k=1}^{j}(u_k - d_k) &\ge 0, \\ \sum_{k=1}^{j}(u_k - d_{k-1}) &\le p - 2,\end{aligned} \qquad j = 1, \ldots, \mathcal{N}, \qquad d_0 \equiv 0. \tag{8}$$

We refer to any 2–by–$\mathcal{N}$ matrix of non-negative integer elements which satisfy conditions R1–R3 above, as *admissible*. The point is that one can always use the elements of such a matrix to uniquely construct *a* path. However, the restrictions on the elements of an admissible matrix translate to the restrictions satisfied by a path. For the rest of this work, the term *matrix* will be used to imply an admissible interpolating matrix.

3.2.2 From matrices to paths

Since, as explained above, an admissible interpolating matrix simply encodes the ordered sequences of up- and down-vertices, it can be used to uniquely construct an admissible path, and the map from paths to matrices, as above, is a bijection.

3.2.3 The weight of a path

Given the matrix $M(P)$ corresponding to path P, the weight, $W(P)$, of the latter–which was introduced in §2.1–can be computed as

$$\begin{aligned}W(P) = \ &\frac{1}{2}\left(\sum_{k=1}^{u_1} k + \sum_{k=1}^{d_1}(1 + u_1 + k) + \sum_{k=1}^{u_2}(2 + u_1 + d_1 + k)\right. \\ &\left. + \sum_{k=1}^{d_2}(3 + u_1 + d_1 + u_2 + k) + \ldots\right).\end{aligned} \tag{9}$$

Using restriction R1 on the elements of $M(P)$, and defining the number D by

$$D \equiv \sum_{j=1}^{\mathcal{N}} u_j = \sum_{j=1}^{\mathcal{N}} d_j, \tag{10}$$

we can rewrite (9) as

$$W(P) = D(D+1) + \sum_{j=1}^{\mathcal{N}} \left[\sum_{k=j+1}^{\mathcal{N}} u_k + \sum_{k=j+1}^{\mathcal{N}} d_k \right]. \tag{11}$$

This form of the weight function will be useful later. Basically, we will translate it directly into a Ferrers graph of a partition of $W(P)$.

3.3 A bijection between matrices and partitions

Given an admissible matrix $M(P)$, with a weight $W(P)$ given by equation (11), we construct a Ferrers graph of the number $W(P)$ as follows:

3.3.1 From matrices to partitions

P1 We translate the term $D(D+1)$ to a rectangle of nodes, which has D rows and $(D+1)$ columns. This is basically a Durfee square, plus an extra column of length D, of some Ferrers graph. Let us refer to this as a *Durfee rectangle*. In the following, we generate the rest of the graph from the other two terms in (9).

P2 The lengths of the $(D+j)$-th row (below the Durfee rectangle constructed above) and the $(D+j+1)$-th column (to the right of the Durfee rectangle) are given by

$$\begin{aligned} r_{D+j} &= \sum_{k=j+1}^{\mathcal{N}} u_k, & j &= 1, \ldots, \mathcal{N}, \\ c_{D+j+1} &= \sum_{k=j+1}^{\mathcal{N}} d_k, & j &= 1, \ldots, \mathcal{N}. \end{aligned} \tag{12}$$

We note that P1 and P2 only make sense provided D is defined as in (10). Hence in our construction of partitions we have implicitly used condition R1 on admissible matrices. This also implies the extra equation

$$u_1 = D - r_{D+1}. \tag{13}$$

needed to invert the transformation from $M(P)$ to $W(P)$. We also note that, from (10), $j = 0$ in the first equation in (12) gives the dimension of the side of the Durfee square: D, which is already constructed in step P1 above. similarly, $j = 0$ in the second equation in (12) gives the right-most column of the Durfee rectangle, which is also already constructed.

The above construction uniquely generates a Ferrers graph, which has a Durfee rectangle of dimensions D–by–$(D+1)$. Since the total number of nodes of a Ferrers graph is

$$D^2 + \sum_{j>D} (r_j + c_j), \tag{14}$$

it follows from (11) that (12) indeed defines a partition of the number $W(P)$.

Next, we have to show that the Ferrers graph generated, as explained above, are indeed what we are looking for. In other words, it has to satisfy the conditions F1 and F2 of §2.2.

3.3.2 The partitions obtained are admissible

First, we consider the conditions R1 and R2. Setting $j = \mathcal{N}$ in (12) we obtain $r_{D+\mathcal{N}} = c_{D+\mathcal{N}+1} = 0$. However, from R1 and R2 we get $N + D = m$, yielding $r_m = c_{m+1} = 0$. Thus, via (12), each admissible matrix defines a Ferrers graph which has maximally $(m-1)$ rows and m columns. Concluding, restrictions R1 and R2 on the admissible matrices imply the restriction F1 on admissible partitions.

Next, restriction R3, as formulated in equation (8), can be translated into conditions on the rows and columns of the Ferrers graph as follows:

$$\sum_{k=1}^{j}(u_k - d_k) = \sum_{k=j+1}^{\mathcal{N}}(d_k - u_k) = c_{D+j+1} - r_{D+j} \geq 0, \tag{15}$$

$$\sum_{k=1}^{j}(u_k - d_{k-1}) = \sum_{k=j+1}^{\mathcal{N}}(d_{k-1} - u_k) = c_{D+j} - c_{D+\mathcal{N}} - r_{D+j} \leq p - 2, \tag{16}$$

for all $j = 1, \ldots, \mathcal{N}$. The length of the j-th row (column) $j = 1, \ldots, D$ is $D + x_j$ ($D + y_j$) with x_j (y_j) the smallest integer such that $c_{D+x_j+1} < j$ ($r_{D+y_j+1} < j$), which implies $c_{D+x_j} > c_{D+x_j+1}$ ($r_{D+y_j} > r_{D+y_j+1}$). We thus find that the hook differences on the 0-th diagonal are ≥ 1, and on the $(p-2)$-th diagonal are ≤ 0.

We have therefore mapped admissible matrices on Ferrers graphs of admissible partitions.

3.3.3 From partitions to matrices

The above map is invertible since the dimensions of the building blocks of an admissible Ferrers graphs, namely the Durfee square, and the rows and columns adjacent to it, can be translated to elements of an admissible matrix using the same equations as above.

Finally, we need to show that the restrictions on the admissible Ferrers graphs translate correctly to the restrictions on admissible matrices.

3.3.4 The matrices obtained are admissible

M1 Restriction R1 is trivially satisfied

M2 Restriction R2 can always be satisfied, since it basically amounts to adding extra columns with zero entries to the matrix until $D + \mathcal{N} = m$ is satisfied

M3 Restriction R3 is satisfied given that we start from an admissible partition.

This completes our bijection between paths and partitions. As a consequence of that, the corresponding generating functions–as obtained in [6, 7]–are equal, which implies (1).

Before concluding we note the following simple graphical way to construct the Ferrers graph of an admissible partition from an admissible matrix.

- Draw a partition of $2\mathcal{N}$ rows, $r_j = 2\mathcal{N} + 1 - j$, $r_{j+N} = \mathcal{N} - j$, $j = 1, \ldots \mathcal{N}$.

- Multiply the j-th row $d_{\mathcal{N}-j+1}$ times and the j-th column $u_{\mathcal{N}-j+1}$ times, $j = 1, \ldots, \mathcal{N}$.

Carrying out both steps result in the admissible partition. For the partition of Fig. 2 this is shown in Fig. 3. We thus find that the path of Fig. 1 corresponds to the partition of Fig. 2. The reverse of this statement is true provided $m = 13$.

4 Discussion

In this letter, we established a bijection between a class of restricted lattice paths and a class of partitions with prescribed hook differences. As a result of this, we obtain a bijective proof a Melzer's polynomial identities related to the Virasoro characters $\chi_{1,1}^{(p,p+1)}(q)$.

We restricted ourselves to lattice paths confined to $0 \leq y \leq p-1$, starting in $(0,0)$ and ending in $(2m,0)$. We expect that a generalization of the above bijection exists, such that one can map lattice paths starting confined to this same strip, but starting in $(0,s-1)$ and ending in $(M,r-1)$, to partitions with hook differences on the $(1-r)$-th diagonal $\geq r-s+1$ and on the $(p-r-1)$-th diagonal $\leq r-s$. If so, this would amount to a bijective method of proof for all polynomial $\chi_{r,s}^{(p,p+1)}(q)$ character identities [1].

Acknowledgements

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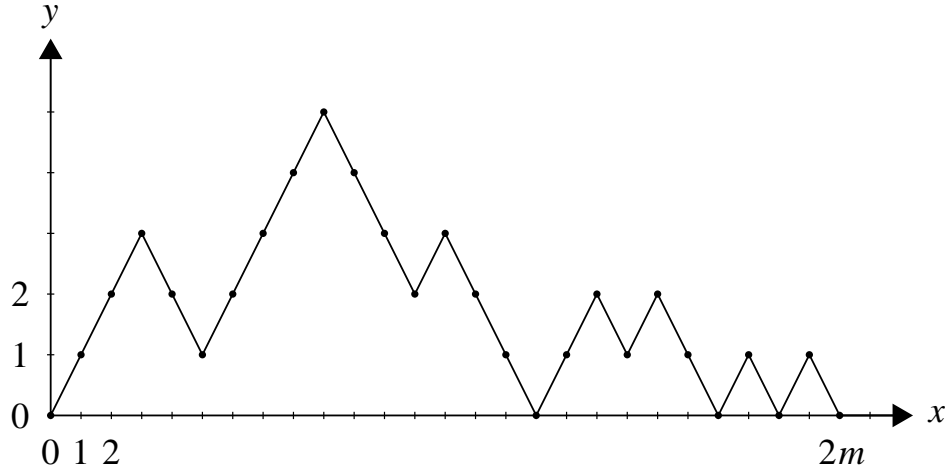
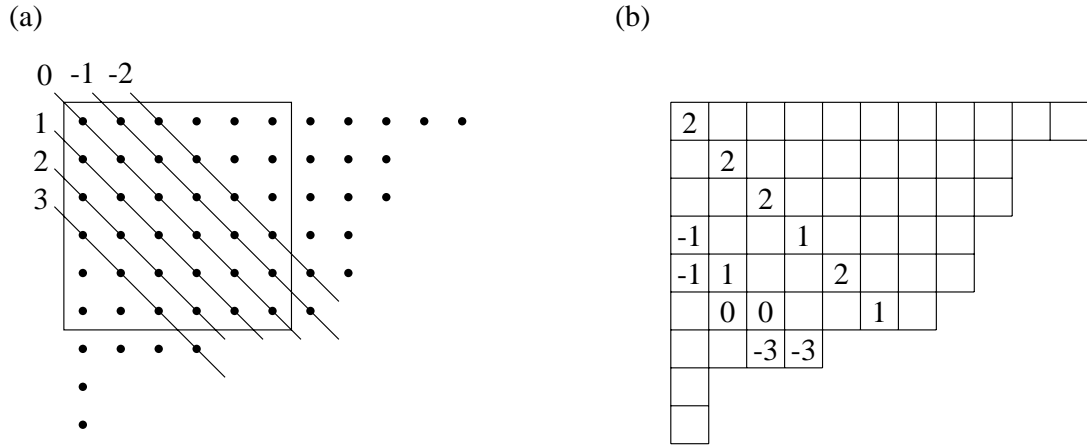Figure 1: Example of a restricted lattice path for $p \geq 6$.

(a)

(b)

Figure 2: (a) Ferrers graph of the partition 11,9,9,8,8,7,4,1,1. The Durfee square has size $D = 6$. (b) Young representation of the partition, with listing of hook differences on the 0-th, 3-th and 4-th diagonal.
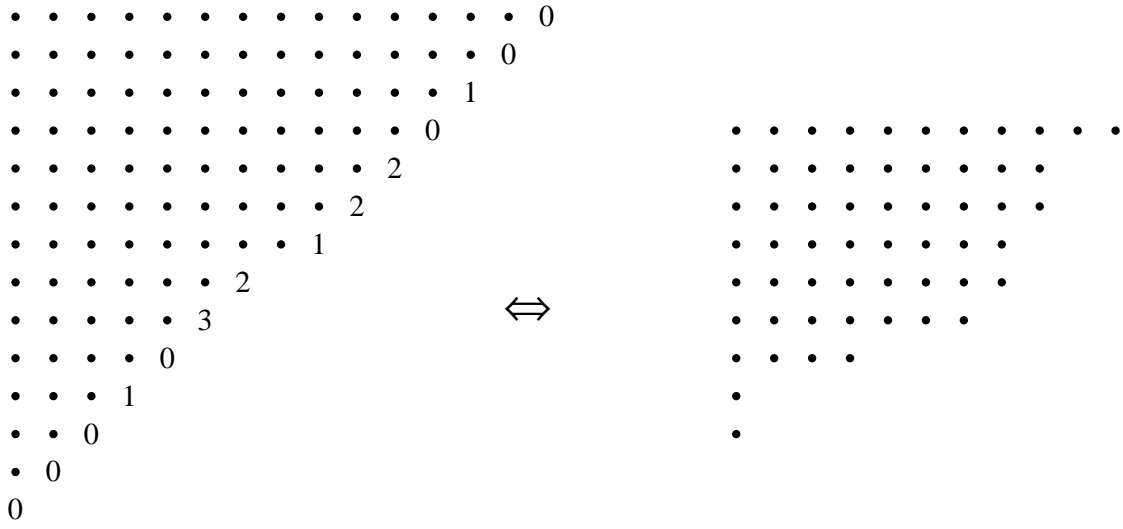
Figure 3: The construction of a Ferrers graph out of the admissible matrix $M = \begin{pmatrix} u_1 & u_2 & \dots & u_{\mathcal{N}} \\ d_1 & d_2 & \dots & d_{\mathcal{N}} \end{pmatrix} = \begin{pmatrix} 2 & 3 & 0 & 1 & 0 & 0 & 0 \\ 1 & 2 & 2 & 0 & 1 & 0 & 0 \end{pmatrix}$. Clearly, from the resulting graph and the value of m one can easily reconstruct the "pre-graph". Hereto we recall that the dimension D of the Durfee square is $\sum u_j = \sum d_j$ and that $N = m - D$.